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THE MEAN FLOW CHARACTER OF TWO-PHASE FLOW EQUATIONS

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Abstract—This study examines the nature of the one-dimensional mean motion description for two-phase flows. It is conjectured that the unstable wave growth in a streaming two-phase flow with unequal phasic velocities is a result of the failure to model the correlations of the fluctuating velocity components in the momentum equations. A general functional form for these velocity correlations is derived based upon invariance and dimensional arguments. Some speculative closure models using this functional form are derived and it is shown that reasonable forms of this closure model do in fact lead to a stable mean motion description.

INTRODUCTION

In many areas of two-phase flow it is desirable to have model equations that describe the behavior in terms of quantities averaged in either space or time. There have been many formally exact derivations of these averaged equations (Ishii 1975; Delhaye 1979). Like their single-phase turbulent counterparts, however, the real difficulty lies in choosing the appropriate closure formulas for each of the many interactions and fluctuation terms introduced by this formal averaging process.

The standard closure assumptions neglect the terms representing the correlations of fluctuating velocity components in the mean momentum equations. In the single-phase turbulent counterpart these are the Reynolds stress terms. It is well known that appropriate closure models for the Reynolds stress terms is the essence of the problem of obtaining a realistic mean motion description of turbulent flows. In particular, if one were to make the assumption that the Reynolds stress terms could be neglected in the mean flow equations then the mean flow equations would exhibit the same instability as was present in the local laminar flow equations. If the Reynolds stress terms were neglected the very purpose of forming the mean flow equations—to average out the unstable local flow—would not have been achieved.

It is well known that the standard closure assumption for the two-phase flow equations neglecting the correlations of fluctuating velocities results in average mean flow equations that manifest Helmholtz flow instabilities. By analogy with the turbulent flow situation this manifestation of the local Helmholtz instability in the mean flow equations may be the result of a failure to include an appropriate model for the fluctuations of velocity.

These instabilities in the mean flow equations have usually been treated in the context of a characteristic analysis and related to the ill-posed nature of the mean model equations.

In this paper we will give evidence to support the conjecture that the fundamental problem giving rise to the mean flow instabilities in two-phase flow is not primarily related to the short wavelength response (characteristics) but may be instead a result of the failure to model the correlations of the velocity fluctuations in the mean momentum equations.

2. THE BASIC MODEL EQUATIONS

We begin with the mean motion equations for two-phase flow as derived by Delhaye (1979) and Ishii (1975). These equations formally describe a very wide class of possible flows. For this study we shall make certain simplifying assumptions which reduce this general formulation to one that is more manageable, yet one that retains the basic physics that results in the well known two-phase flow instabilities (Ramshaw 1978; Hetsroni 1982; Gidaspow 1974).

First, we consider phases for which both fluids may separately be assumed incompressible. Second, we consider flows in which thermal processes are unimportant, that is, flows with no heat or mass transfer between the fluids. Third, we shall assume that viscous forces may be neglected in the bulk phases. In addition, we will primarily be concerned with stratified flows. At the interface where there is relative motion a boundray layer must develop in which viscous forces are present. But it is felt that the existence of this high shear layer will primarily affect only the high-frequency short wavelength phenomena. Although viscous effects will generally be neglected we will keep in mind that the inclusion of such effects will modify the short wavelength dispersion analysis.†

The spacially averaged model equations for the one-dimensional situation are rigorously derived in appendix A. These averaged equations become‡

$$\frac{\partial \alpha_G}{\partial t} + \frac{\partial \alpha_G v_G}{\partial x} = 0 \quad , \tag{1}$$

$$\frac{\partial \alpha_L}{\partial t} + \frac{\partial \alpha_L v_L}{\partial x} = 0 \quad , \tag{2}$$

$$\alpha_G \rho_G \left(\frac{\partial v_G}{\partial t} + v_G \frac{\partial v_G}{\partial x} \right) + \frac{\partial \alpha_G \rho_G \overline{v'_G v'_G}}{\partial x} = - \frac{\partial \alpha_G P_G}{\partial x} + P_{GI} \frac{\partial \alpha_G}{\partial x} \quad , \tag{3}$$

$$\alpha_L \rho_L \left(\frac{\partial v_L}{\partial t} + v_L \frac{\partial v_V}{\partial x} \right) + \frac{\partial \alpha_L \rho_L \overline{v'_L v'_L}}{\partial x} = - \frac{\partial \alpha_L P_L}{\partial x} + P_{LI} \frac{\partial \alpha_L}{\partial x} \quad , \tag{4}$$

$$\alpha_G + \alpha_L = 1 \quad , \tag{5}$$

where v and P represent the mean phasic velocity and pressure, ρ is the constant phasic density and α represents the void fraction. P_{GI} and P_{LI} represent the average phasic interface pressures. v'_G and v'_L represent the deviation of the local velocities from their mean values, $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$ represent the averages of the deviations squared. In [1] through [5] the shear drag terms at the wall and interface have been neglected. If these terms are included and modeled with algebraic equations the resulting friction terms, because of their algebraic nature, primarily affect long wavelength phenomena. The arguments used in the following development concern the medium wavelength response of the above equations so these frictional terms are neglected. A differential model for these drag forces taking into account transient drag forces (commonly called added mass effects) has been studied (Lyczkowski *et al.* 1978). For the stratified flow situation these transient drag forces are extremely small and the analysis in Lyczkowski *et al.* (1978) shows that their inclusion does not remove the basic instability present in [1] through [5]. For the above reasons the interface and wall drag terms are neglected in [1] through [5].

The usual two-phase flow equations are obtained from the above equations by making the following additional assumptions:

$$\text{Assumption 1} \qquad P_{GI} \simeq P_G \simeq P_L \simeq P_{LI} \, (\triangleq P) \quad , \tag{6}$$

$$\text{Assumption 2} \qquad \overline{v'_L v'_L} \simeq 0 \quad , \overline{v'_G v'_G} \simeq 0 \quad . \tag{7}$$

When these two assumptions are invoked the resulting system predicts unstable growth for all wavelengths in a simple streaming flow with $v_G \neq v_L$.

3. THE PURPOSE OF AVERAGING

The purpose of averaging the local flow equations quite simply put is to remove the local flow fluctuations and obtain model equations representative of the mean flow behavior. If one wanted to follow the local flow the unaveraged equations should be used. This local

† See further comments in section 3.

‡ Although a spacial averaging process has been used to obtain these model equations the form of the model equation is basically the same for all types of averaging processes, i.e. spacial, time, space-time, ensemble. Later in this paper when we need to evaluate averaged quantities from local variables we will use spacial averaging for the derivation although similar results can be obtained using other forms of averaging.

problem is even more complex in two-phase flow with the multiple internal interfaces than it is in single-phase flow. Let us consider two examples to clarify these concepts.

Example 1—Two-phase flow

Consider the classical Helmholtz problem for two fluids in plane motion confined between plates a distance H apart. This two-dimensional flow can be exactly analyzed for our case, i.e. incompressible, inviscid flow without mass transfer. A summary of this analysis is contained in appendix B. As can be seen from the dispersion relationship [B-5] this exact analysis predicts unstable wave growth (complex ω) for all wavelengths when $v_G \neq v_L$. The growth rate becomes unbounded as the wave number $k \to \infty$, which makes this problem ill-posed (Richtmeyer 1967). This ill-posed character of the two-dimensional exact solution can be removed by the inclusion of any physically realistic model that adds a restoring or damping force for small wavelengths. The inclusion of surface tension (Ramshaw 1978) or viscosity (Richtmeyer 1967) adds such a term to the equations and results in a well-posed Helmholtz problem.† If we assume that such terms have been included in the analysis, [B-5] will still predict instabilities for all but the shortest wavelengths. The Helmholtz problem is a linearized perturbation analysis that predicts the local motion of small disturbances superimposed on the parallel streaming flow. In reality, as the wave growth predicted by this two-dimensional analysis proceeds in time nonlinear effects begin to manifest themselves and the growing waves are stabilized with some resulting bounded amplitude. This motion may be that of an organized wave motion (solitary waves) or unorganized wave motion (turbulent wave height fluctuations). Both are observed in practice (Tritton 1977). In either case the organized/unorganized wave motion with wavelength of the order of the channel height H cannot be followed with any accuracy using averaged one-dimensional equations. In fact, the *very purpose* of forming the averaged equations (instead of using the local multidimensional equations) was to eliminate the fluctuations with wavelengths of order H and to follow only the mean motion.

A brief comment is in order at this point to define the relative terms short, medium and long wavelength. For our discussion short wavelengths are of the order of the channel height, i.e. wavelengths less than three or four H. The control volume used to develop the averaged mean flow equations is considered to have length H (see appendix A). Such short wavelength phenomena are clearly not modeled in the averaged equations. Medium wavelengths are of the order 4 to 15 times the channel height. It is expected that wavelength phenomena of this order should be reasonably modeled by our mean motion equation. Wavelengths larger than 15–20 H will be considered long. For our incompressible zero mass transfer situation the relationship between wavelength and frequency can be made quite explicit. This relationship is discussed in appendix A.

Example 2—Multidimensional single phase

An analogous situation holds in single-phase multidimensional flow at the onset of transition from laminar to turbulent flow. In this case the local linearized equations predict a growing instability for some finite wavelength. This instability eventually gives rise to unorganized wave motion (turbulence). If one wants to develop model equations for the mean motion of a turbulent flow field he must average the local equations. It is very important in this averaging process to include a model for the fluctuation terms $\overline{v_i'v_j'}$ (commonly called the Reynolds stress terms). In fact, this is the *essence* of obtaining correct mean motion equations. If (as no one does) the velocity correlations, $\overline{v_i'v_j'}$, had been neglected in the closure of the mean motion equations then the averaged equations for the mean motion would still contain the same instability as was present in the local equations—a totally unacceptable procedure.

The multidimensional turbulent example is quite analogous to the stratified flow example if we remember that the Helmholtz problem with $v_G \neq v_L$ can be thought of as a viscous flow problem with a shear layer of extremely small width d (the interface) and extremely large transverse velocity gradients. If the Helmholtz problem is examined from this point of view then the linearized stability analysis in appendix B is somewhat parallel

† The analysis cannot be carried out exactly for viscosity but the exact solution including surface tension can be obtained; see Ramshaw (1978).

to the stability analysis performed for the onset of turbulent flow using the inviscid form of the Orr–Sommerfeld equation (Schlichting 1958). Although the inviscid Orr–Sommerfeld dispersion equation cannot be used to obtain a critical Reynolds number for the onset of turbulence, it is possible with this type of analysis to answer the question as to whether a given laminar flow is stable or not (Schlichting 1958, pp. 388–389). Using the inviscid stability equation [B-4] for the Helmholtz problem then says that the instability predicted for any $v_G \neq v_L$ is an analysis for the onset of turbulent (organized or unorganized) wave motion.

The above discussion leads one to believe that a proper averaged form of the local flow equations for two-phase flow must contain a model for the velocity correlations, $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$, if one wants to properly follow the mean motion. If one neglects the $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$ terms in [3] and [4] then it is expected that the averaged mean flow equations will still contain the same flow instabilities that were present in the local equations before the averaging was done. This is exactly what is observed in the averaged two-phase flow equations if the velocity correlation terms are neglected. This instability in the mean two-phase flow equations for a uniform streaming flow is well known.

Before we proceed with our closure assumptions for $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$ we want to make a few remarks about the closure assumptions needed for pressure. If we examine [1]–[5], neglecting the velocity fluctuations, then we must make some assumptions for the pressure terms. If assumption 1 is made then it is well known that the basic averaged two-phase flow equations exhibit the same Helmholtz instability as was exhibited in the local two-dimensional unaveraged equations.

Some authors have made different assumptions regarding the pressure terms. Ramshaw & Trapp (1978) have included a surface tension, σ, contribution to give a model for $P_{GI} - P_{LI}$ (with $P_G = P_{GI}$ and $P_L = P_{LI}$). Their analysis clearly shows that the local Helmholtz instability still remains in the averaged equations for all but the shortest wavelengths where σ removes the unbounded growth. Hence this pressure assumption also leads to averaged model equations in which the local Helmholtz instabilities remain.That is to say, the very purpose for which averaging is generally performed (to remove local fluctuations and follow the mean motion) was not achieved with this system. This was realized in Ramshaw (1978) where the authors state: "In the present paper, attention has been implicitly restricted to the problem of obtaining a satisfactory system of equations to describe the *Instantaneous* motions in separated two-phase flow," and "[the analysis]...would clearly not apply if the objective were to obtain an equation system which does not attempt to describe small amplitude disturbances correctly, but seeks instead to describe the *average* behavior of a 'fully developed' two-phase flow in which apparently random motions reminiscent of turbulence appear to be superimposed on a slowly varying deterministic mean motion."

A different assumption for the phasic pressures has been made in Ransom (1984). Here the authors use transverse momentum considerations (leading to additional differential equations or algebraic conditions) to relate P_L to P_G. If a linearized stability analysis of the resulting model equations is performed one again finds that except for small wavelengths the local Helmholtz instability is still manifested in their averaged mean motion model equations. Hence this system again fails to model the mean behavior but retains the instability responsible for the local fluctuations that were supposed to have been averaged out.

A model for the phasic pressures can be developed which includes the effect of a transverse gravity head. If such a model is included then one obtains the standard single phase pressure terms plus a derivative term proportional to $\partial\alpha/\partial x$. It is well known that such a tem will stabilize a streaming flow with $v_L \neq v_G$ for moderate values of the velocity difference. This effect is clearly not present in a vertical flow. Hence there are several regimes (larger relative velocity differences or vertical flows) where the inclusion of gravity head effects fails to give a model that manifests stable mean motions. It should be noted that for larger relative velocities the above-mentioned gravity head model fails to stabilize any wavelengths. In particular short wavelengths that should have been filtered out by the averaging process still remain and manifest themselves in the mean motion description.

This extended digression concerning the different assumptions that have been made regarding the pressure terms in [1]–[5] has been included to show that the basic problem (local flow instabilities manifesting themselves in the averaged mean motion equations) is not always removed by better pressure approximations. This is further evidence to support the conclusion that the proper mean motion equations will only be obtained if a realistic (nonzero) model for $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$ is included in the averaged mean motion equations [3] and [4].

4. THE REQUIRED FORM OF THE CLOSURE TERMS

A model for the velocity correlation terms in [3] and [4] is extremely difficult to derive. In fact, it is probably as difficult as was the development of models for the Reynolds stress terms in single-phase turbulent flows. In the Reynolds stress models it is relatively easy to convince oneself that a Reynolds stress proportional to the mean shear gradients is a reasonable first approximation†. This assumption can be motivated by physical arguments. The more difficult problem is to determine the proportionality "constant," i.e. the eddy viscosity. Several different models have been proposed and used for the eddy viscosity (Schlichting 1958). In this section we would like to motivate a model for the velocity correlations, $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$, that appear in the mean flow equations [3] and [4].

In the local two-dimensional Helmholtz problem the local instability is generated by a mean relative velocity $v_G - v_L$ that is nonzero. The local instability is nonexistent if $v_G - v_L$ is zero. Since there are no mean gradients in the steady flow considered any algebraic closure model for $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$ must be a function of v_G, v_L, ρ_G, ρ_L, α_G and possibly some additional parameters characterizing the local scale. Now models for $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$ must vanish when $v_G - v_L$ equals zero. It can be shown by invariance arguments (Truesdell 1956), that $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$ can only depend upon v_G and v_L in the combination $(v_G - v_L)$.These arguments alone require the modeling‡

$$\overline{v'_G v'_G} = f_G (v_G - v_L)^2 \quad , \qquad \overline{v'_L v'_L} = f_L (v_G - v_L)^2 \quad , \tag{8}$$

where f_G and f_L must be dimensionless functions of $(v_G - v_L)/v_M$, ρ_G/ρ_L, α_G, and (possibly) local scale parameters denoted by LS. Here v_M is some representation of a mean velocity, say $(\alpha_G \rho_G v_G + \alpha_L \rho_L v_L)/(\alpha_G \rho_G + \alpha_L \rho_L)$§.

Although the form in [8] is established by very simple invariance arguments it can also be motivated by more physical arguments parallel to those commonly used in turbulence modeling. In particular, if we think of the local Helmholtz instability as generated by a thin shear layer (of width d) with a larage transverse velocity gradient then by analogy with the turbulent flow situation the velocity fluctuations may be assumed proportional to the velocity gradient at the interface. This gives for $\overline{v'_G v'_G}$

$$\overline{v'_G v'_G} = f'_G \left(\frac{v_G - v_L}{d}\right) \quad , \tag{9}$$

where f'_G is a function with the same arguments as f_G and d represents the local scale, i.e. it corresponds to the LS argument in f_G. Since $\overline{v'_G v'_G}$ is nonnegative and $v_G - v_L$ can be either positive or negative dimensional arguments applied to [9] require it to take the form shown in [8] with d absorbed in f_G as LS.

With [8] established as the basic formulation for the velocity correlations we are now at a point in our development that parallels the multidimensional turbulent model for the Reynolds stress,

$$\overline{v'_i v'_j} = \epsilon \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \quad . \tag{10}$$

† All the algebraic closure models for turbulence use this assumption; Prandtls mixing length theory, Taylor's vorticity transfer theory, etc. We will only be considering algebraic closure model in this paper.

‡ If we wanted to include the effects of the wall then invariance arguments would also allow dependence upon v_G^2 and v_L^2. For the inclusion of such effects in single phase pipe flow; see Slattery (1972). The nature of the problem is associated with the interface behavior and the shear instability generated there so wall effects have been neglected.

§ A more appropriate value may be $(\alpha_G \rho_L v_L + \alpha_L \rho_G v_G) / (\alpha_G \rho_L + \alpha_L \rho_G)$ which is the speed at which the local Helmholtz waves travel, i.e. the kinematic wave speed.

We have the basic form for the velocity correlations, proportional to $(v_G - v_L)^2$, but we have two unknown "eddy viscosities" f_G and f_L. It is felt that a valid form for these dimensionless functions could only be firmly established by hypotheses backed up by experimental measurements or numerical experiments.† As in multidimensional turbulent flow there may be several models for f_G and f_L that give basically the same mean motions. Since the kinematic wave speeds in the mean equations including the model in [8] will depend upon f_G and f_L it is felt that f_G and f_L could be determined by experiments on the mean motion propagation properties without many detailed local experimental measurements. Although this paper is not experimental in nature, approximate forms for f_G and f_L will be suggested based upon physical/analytic considerations.

5. SOME SPECULATIVE CLOSURE MODELS

A. A closure model using the local Helmholtz velocities for a guideline

If we return to the local Helmholtz instabilities analyzed in appendix B, we can make the plausible argument that these instabilities grow in time but become bounded by nonlinearities. One might expect that the nonlinearities present in this bounded wave motion would give rise to small corrections to the linearized wave profiles seen in appendix B. In particular we could form $\overline{v'_G v'_G}$ using the profiles in appendix B to give a first estimate for f_G and f_L. This is now done.

This development will be carried out for the gas phase. The final parallel results for the liquid phase will then be recorded. The plan is as follows:

1. Use the local wave motion in [B-1] to form $v'_G v'_G$ for a particular wave number k.
2. Average the local fluctuating velocity products over a spacial region the length of which is $2\pi/k$.
3. Write the averaged $\overline{v'_G v'_G}$ in the form displayed in [8].
4. Examine the resulting form of the coefficient of $(v_G - v_L)^2$ to obtain a first approximation to f_G.

From [B-1] it is easy to see that $v'_G v'_G$ for any k is

$$\overline{v'_G v'_G} = \left[H\alpha_0 i\,(\omega + v_G k)\frac{\cosh\,[k\,(y - H\alpha_G)]}{\sinh\,[kH\alpha_G]}\,e^{i(\omega t + kx)}\right]^2 \quad . \tag{11}$$

Averaging $v'_G v'_G$ in [11] over the region $0 \le y \le H\alpha_G$, $x_0 \le x \le x_0 + 2\pi/k$, gives

$$\overline{v'_G v'_G} = \alpha_0^2\,e^{\pm 2\omega_I t}\frac{\rho_L^*}{2\alpha_G^2\,(\rho_L^* + \rho_G^*)}\,(v_G - v_L)^2 \quad , \tag{12}$$

where α_0 is the initial wave height disturbance, ω_I is the complex part of ω in [B-5] and

$$\rho_L^* = \rho_L \coth\,(kH\alpha_L) \quad , \qquad \rho_G^* = \rho_G \coth\,(kH\alpha_G) \quad . \tag{13}$$

Because $\overline{v'_G v'_G}$ is derived from the unstable local motions the velocity correlation in [12] manifests the $e^{+2\omega_I t}$ growth factor. We have assumed that the nonlinearities eventually make this term bounded with small resultant changes in the velocity profiles. This being the case we have for f_G

$$f_G = A_G \frac{\rho_L^*}{2\alpha_G^2\,(\rho_L^* + \rho_G^*)} \quad , \tag{14}$$

where A_G is some dimensionless function of the same arguments as f_G.

We have argued throughout the whole paper that local fluctuations with wavelengths of the order of the channel thickness H must be averaged out in the mean flow equations.

† The state of the art relative to numerical simulations of stratified flows with interface tracking makes a numerical determination of $\overline{v'v'}$ very reasonable at the present time.

This implies that f_G in [14] should be used with k of the order $2\pi/H$. Using [14] we have for f_G

$$f_G = A_G \frac{\rho_L \coth(2\pi\alpha_L)}{2\alpha_G^2}(\rho_L \coth(2\pi\alpha_L) + \rho_G \coth(2\pi\alpha_G)) \qquad [15]$$

when $k = 2\pi/H$. The void fraction dependence in [15] is complicated by the coth functional dependence. We can approximate this dependence by considering two special cases: (1) $\alpha_G = \alpha_L = 1/2$, and (2) $\alpha_G \to 0$, $\alpha_L \to 1$. For case (1) we obtain from [15]

$$f_G = \frac{A_G}{2\alpha_G^2}\left(\frac{\rho_L}{\rho_L + \rho_G}\right) \quad . \qquad [16]$$

For the limiting case (2) we have $\coth(2\pi\alpha_L) \cong \coth(2\pi) \cong 1$ and $\coth(2\pi\alpha_G) \cong 1/2\pi\alpha_G$; hence from [15] we obtain

$$f_G = \frac{A_G}{2\alpha_G^2}\left(\frac{2\pi\alpha_G\rho_L}{2\pi\alpha_G\rho_L + \rho_G}\right) \quad .$$

Remembering that we are considering the limiting case $\alpha_G \to 0$ this reduces to†

$$f_G = A_G \frac{\pi}{\alpha_G^2}\left(\frac{\alpha_G\rho_L}{\rho_G}\right) \quad . \qquad [17]$$

Now a general form for f_G that includes both of these cases, i.e. [16] for $\alpha_L = \alpha_G$ and [17] for α_G small, is

$$f_G = \frac{A_G}{\alpha_G^2}\left(\frac{\alpha_G\rho_L}{\alpha_G\rho_L + \alpha_L\rho_G}\right) \quad , \qquad [18]$$

where the 1/2 or π factor has been absorbed in the void fraction dependence of A_G.

This procedure then gives the following models for the velocity correlations in [3] and [4]:

$$\alpha_G\rho_G \overline{v'_G v'_G} = A_G \frac{\rho_G\rho_L}{\rho_M}(v_G - v_L)^2 \, , \qquad [19]$$

$$\alpha_L\rho_L \overline{v'_G v'_L} = A_L \frac{\rho_G\rho_L}{\rho_M}(v_G - v_L)^2 \quad , \qquad [20]$$

where $\rho_M = \alpha_G\rho_L + \alpha_L\rho_G$ and the A's are dimensionless functions of $(v_G - v_L)/v_M$, ρ_G/ρ_L, α_G, and possibly LS. If $\overline{v'_G v'_G}$ and $\overline{v'_L v'_L}$ are bounded for all void fractions, as in reality they must be, then from [19] A_G must vanish as $\alpha_G \to 0$ and from [20] A_L must vanish as $\alpha_L \to 0$. If an alpha is explicity factored out of each of the A terms, i.e. $A_G = \alpha_G E_G$, $A_L = \alpha_L E_L$, we then have

$$\alpha_G\rho_G\left[\frac{\partial v_G}{\partial t} + v_G\frac{\partial v_G}{\partial x}\right] + \frac{\partial}{\partial x}\left[\alpha_G E_G\left(\frac{\rho_G\rho_L}{\rho_M}\right)(v_G - v_L)^2\right] + \alpha_G\frac{\partial P}{\partial x} = 0 \quad , \qquad [21]$$

† The α_G^2 dependence in [17] in the denominator implies that A_G must have an α_G dependence of at least order one.

$$\alpha_L \rho_L \left[\frac{\partial v_L}{\partial t} + v_L \frac{\partial v_L}{\partial x} \right] + \frac{\partial}{\partial x} \left[\alpha_L E_L \left(\frac{\rho_G \rho_L}{\rho_M} \right) (v_G - v_L)^2 \right] + \alpha_L \frac{\partial P}{\partial x} = 0 \quad , \tag{22}$$

as the form of the momentum equations including an approximate model for the velocity fluctuation correlations.

At this point we have given a tentative model for the fluctuating velocity terms. It is to be noted that this model involves two unknown functions E_G and E_L. Hence, at this point we are still unable to analyze the character of the mean flow equations. An explicit form for the correlation functions E_G and E_L will be suggested in section B, [28]. It will then be possible to perform a stability analysis for the mean motion equations. The result is contained in [29].

B. A further specification assuming stable mean motion equations

Our whole analysis has been motivated by the fact that the averaged mean motion equations with a proper model for the velocity fluctuations should predict a stable mean motion without any manifestation of the local Helmholtz instability. To confirm this property of the mean motion equations one must perform a dispersion analysis on [1], [2], [21] and [22]. In this dispersion analysis we are examining wave propagation for large scale mean motion waves, not the local wave motions or ripples seen in the Helmholtz problem.

Before proceeding some observations on the long wavelength behavior of the Helmholtz solution are needed. If we consider [B-5] for long wavelengths, i.e. for $\alpha_G kH$ and $\alpha_L kH$ going to zero, then using $\coth(\alpha_G kH) \to 1/\alpha_G kH$ and $\coth(\alpha_L kH) \to 1/\alpha_L kH$ we obtain the dispersion relationship

$$\frac{\omega}{k} = - \frac{(\alpha_G \rho_L v_L + \alpha_L \rho_G v_G)}{\rho_M} \pm \frac{(v_G - v_L)}{\rho_M} \sqrt{- \alpha_G \alpha_L \rho_G \rho_L} \quad , \tag{23}$$

for the propagation of long wavelength local disturbances. Let us think of the Helmholtz problem as an approximation to a streaming flow with two different velocities and a thin shear layer separating them. The Helmholtz analysis shows that the growth rate proportional to the complex part of ω increases as the wavelength decreases. In reality viscous and/or surface tension effects eventually come into play at the very short wavelengths to mitigate the extreme short wavelength growth.

For long wavelengths there will be little growth. If v_M is defined as

$$v_M = \frac{\alpha_G \rho_L v_L + \alpha_L \rho_G v_G}{\rho_M} \tag{24}$$

then [23] can be written as

$$\omega = - k v_M \pm i k \omega_I \quad . \tag{25}$$

The Helmholtz solution for any flow variable S then has the form

$$S = S_0 e^{i(2\pi/L)(x - v_M t)} e^{\pm (2\pi/L) \omega_I t} \, . \tag{26}$$

This represents a wave of length L propagating with speed v_M and small growth rate (for L large) of $2\pi\omega_I/L$.

Now for long wavelength motions, one expects the local Helmholtz solution to properly reflect the properties of the mean equations except for the turbulent dissipation that is present. If viscous effects are included this energy is dissipated into heat and removed from the mechanical motion at the shorter wavelengths. This being the case, we assume that for long wavelength motions the Helmholtz solution is a good approximation to the mean motion equations except for the small growth rate that would be "dissipated" in any mean motion description. One should then expect [23] without the small complex term, i.e. without the second square root term, to be an approximation to the dispersion relationship expected when the mean motion equations are analyzed.

We now return to our mean motion equations [1], [2], [21] and [22]. Based upon the discussion above it seems reasonable to assume that the dispersion analysis of the mean motion equations will predict stable (real) roots for ω with a real part near $-kv_M$. To carry out this dispersion analysis explicit forms for E_G and E_L must be postulated. It can be shown that when $E_L = E_G (\triangleq E)$ the dispersion analysis will lead to [25] with $\omega_I = 0$ if

$$E = \alpha_G \alpha_L \quad . \tag{27}$$

If we generalize the above and assume

$$E_L = E_G = \alpha_G \alpha_L f(\alpha_G, \rho_G/\rho_L, (v_G - v_L)/v_M, LS) \tag{28}$$

with f still an unspecified function of magnitude greater than or equal to one then the dispersion analysis for the mean flow equations [1], [2], [21] and [22] will lead to

$$\frac{\omega}{k} = -v_M \pm \frac{(v_G - v_L)}{\rho_M} \sqrt{(f-1)(\alpha_G \rho_L)(\alpha_L \rho_G)}. \tag{29}$$

It thus appears that there is an infinite variety of functional forms for E_G and E_L that will render the mean motion equations stable with kinematic wave speeds near v_M.

C. General remarks concerning speculative closure models

The velocity correlation models cannot be obtained from any basic physical principle. In the end they are constitutive equations that must be determined by recourse to experimental results.† The above discussion only shows some possible correlation models [19], [20] that are consistent with the physics of the situation. Within these possible models, [28] gives a model that leads to stable mean motion equations in the streaming flow situation. The specific functional forms for E_G and E_L or (f) could be determined by experimental observations on the speed of large scale mean motion waves or numerical simulations to directly calculate $\overline{v'v'}$.

6. CONCLUSION

It has been demonstrated (by arguments parallel to those used in standard multidimensional turbulence modeling) that the flow instabilities present in the mean motion equations of two-phase flow can be explained as a result of the failure to include appropriate closure models for the velocity fluctuations in the momentum equations. If these terms are neglected unstable mean flow equations are exactly what one should expect and what one obtains. Local flow instabilities remain in the mean flow equations used to describe turbulent flow fields if the Reynolds stress terms are neglected.

Although the exact nature of the closure model in [19], [20] (or as further specified in [28]) is very approximate in nature and experimental work must be done in this area, it can be said that the total picture presented is very reasonable. The basic form of the algebraic closure model for the velocity correlations in [8] is on firm theoretical ground. The analysis has shown that an approximate model for the velocity correlations based upon the closure form in [8] can lead to a reasonable mean flow description—a mean flow description that

† Constitutive equations that depend upon the phasic properties and the nature of the flow field.

does not manifest the local Helmholtz instability. The resulting mean flow equations are also hyperbolic in nature and do not manifest any of the problems associated with complex characteristics.

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APPENDIX A

One dimensional spacially averaged two-phase flow equations — A derivation

In this appendix we will derive the one-dimensional equations that result from a spacial averaging of the local fluid equations. This derivation will be carried out for the mechanical case only, i.e. the thermal energy equations will not be developed. The development will also assume that there is no mass transfer between the phases.

The spacially averaged equations can be derived in one of two ways. The local partial differential equations can be spacially intergrated over a control volume. Then by applying the Reynolds transport theorem and the divergence theorem in their various forms these averaged equations can be cast into the desired form. A second method is to start with the balance equations in their integral form. The local equations in a rigorous development are always derived from the integral forms in the first place. We will develop the averaged equations from the integral forms of the basic balance laws written for a moving control volume. In the following, the averaged equations for the liquid phase will be developed. Parallel equations hold for the gas phase.

We consider a stratified flow in a pipe with x denoting the axial coordinate. The pipe has diameter D and cross-section area A. About each point x a control volume $V_L(x,t)$ will be constructed. This control volume will consist of all the liquid in the pipe between the planes $x-L/2$, $x+L/2$ where L is of the order of the pipe diameter. (See figure 1.) The total volume inside the pipe between planes $x - L/2$, $x+L/2$ will be denoted by V. The surface of $V_L(x,t)$ will be denoted by $S_L(x,t)$. $S_L(x,t)$ is further specified as the sum of (1) $S_{LI}(x,t)$ which denotes that part of $S_L(x,t)$ on the interface between the liquid and gas phase; (2) $S_{LL}(x,t)$, which denotes that part of $S_L(x,t)$ occupied by the liquid; and (3) $S_{LW}(x,t)$ which denotes that part of $S_L(x,t)$ bounded by the pipe wall.

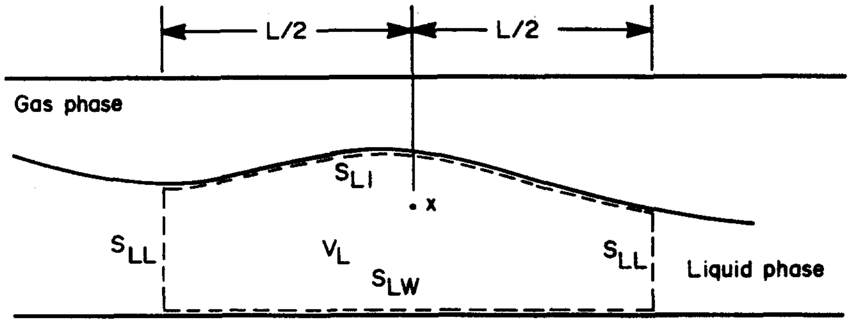

Figure 1. Typical control volume.

The integral form of the conservation of mass principle for the moving control volume $V_L(x,t)$ is

$$\frac{\partial}{\partial t}\left[\int_{V_L(x,t)} \rho^L \mathrm{d}V\right] + \int_{S_L(x,t)} \rho_L(\mathbf{u}_L - \mathbf{u}^S)\cdot \mathbf{n}\,\mathrm{d}S = 0 \quad , \qquad \text{[A-1]}$$

where $\mathbf{u}^s$ is the velocity of the moving control volume surface and $\mathbf{n}$ is the outward unit normal vector. Now on the wall, S_{LW}, we have $\mathbf{u}_L = \mathbf{u}^s = 0$. On $S_{LL}(x,t)$ we have $\mathbf{u}^s = 0$. Since we are considering the situation with zero mass transfer, $\mathbf{u}_L$ equals $\mathbf{u}^s$ on $S_{LG}(x,t)$. With the above values for $\mathbf{u}^s$, [A-1] reduces to

$$\frac{\partial}{\partial t}\left[\int_{V_L(x,t)} \rho_L \,\mathrm{d}V\right] + \int_{S_{LL}(x,t)} \rho_L \mathbf{u}_L \cdot \mathbf{n}\,\mathrm{d}^S = 0 \quad . \qquad \text{[A-2]}$$

Since $\mathbf{n} = (1,0,0)$ on the forward face of S_{LL} and $(-1,0,0)$ on the rearward face we can rewrite the second integral in [A-2] as

$$\int_{S_{LL}\ \text{(forward)}} \rho_L u_{L1}\,\mathrm{d}S - \int_{S_{LL}\ \text{(rearward)}} \rho_L u_{L1}\,\mathrm{d}S \quad .$$

It is a simple matter to show from the definition of a derivative (for a reference, see Slattery (1972)) that

$$\frac{\partial}{\partial x}\left[\int_{V_L(x,t)} \rho_L u_{L1}\,\mathrm{d}V\right] = \int_{S_{LL}\text{(forward)}} \rho_L u_{L1}\,\mathrm{d}S - \int_{S_{LL}\text{(rearward)}} \rho_L u_{L1}\,\mathrm{d}S \quad . \qquad \text{[A-3]}$$

Combining [A-2] and [A-3] we get

$$\frac{\partial}{\partial t}\left[\frac{1}{V}\int_{V_L(x,t)} \rho_L \,\mathrm{d}V\right] + \frac{\partial}{\partial x}\left[\frac{1}{V}\int_{V_L(x,t)} \rho_L u_{L1}\,\mathrm{d}V\right] = 0 \quad , \qquad \text{[A-4]}$$

where we have divided by the constant volume V. Defining the liquid void fraction α_L as V_L/V this can be written as

$$\frac{\partial}{\partial t}(\alpha_L \bar{\rho}_L) + \frac{\partial}{\partial x}(\alpha_L \overline{\rho_L u_{L1}}) = 0 \quad , \qquad \text{[A-5]}$$

where a superposed bar indicates the operation

$$\bar{\psi}(x,t) = \frac{1}{V_L(x,t)} \int_{V_L(x,t)} \psi \mathrm{d}v \,,$$

and is called the phasic average of ψ. For an incompressible liquid [A-5] reduces to

$$\frac{\partial \alpha_L \rho_L}{\partial t} + \frac{\partial \alpha_L \rho_L v_L}{\partial x} = 0 \quad , \qquad \text{[A-6]}$$

where v_L is the average phasic liquid velocity. This completes the derivation of the spacially averaged mass balance equations [1] and [2] of the text.

The intergral form of the liquid momentum equation in the axial direction for the moving control volume $V_L\,(x,t)$ is

$$\frac{\partial}{\partial t}\left[\int_{V_L(x,t)} \rho_L u_{L1}\,\mathrm{d}V\right] + \int_{S_L(x,t)} \rho_L u_{L1}(u_L - u^s)\cdot n\,\mathrm{d}S = \int_{S_L(x,t)} (-P_L n_1 + \sigma^L_{1j} n_j)\,\mathrm{d}S \qquad \text{[A-7]}$$

The body forces are assumed zero and the surface forces consist of a pressure P_L and viscous stresses σ^L_{1j}. Here P_L is the local pressure even though in the text it denotes the phasic average pressure. Using the appropriate values of u_s on the respective subsurfaces of $S_L(x,t)$ we can reduce [A-7]

$$\frac{\partial}{\partial t}\left[\int_{V_L(x,t)} \rho_L u_{L1}\mathrm{d}\,V\right] + \int_{S_{LL}(x,t)} \rho_L u_{L1} u_L\cdot n\,\mathrm{d}S = \int_{S_L(x,t)} (-P_L n_1 + \sigma^L{}_{1j} n_j)\,\mathrm{d}S \quad . \qquad \text{[A-8]}$$

Divide [A-8] by the fixed volume V and apply the same arguments that were used between [A-2] and [A-3] to the second term on the left and we can rewrite [A-8] as

$$\frac{\partial}{\partial t}(\alpha_L \overline{\rho_L u_{L1}}) + \frac{\partial}{\partial x}(\alpha_L \overline{\rho_L u_{L1} u_{L1}}) = \frac{1}{V}\int_{S_L(x,t)} (-P_L n_1 + \sigma^L_{1j} n_j)\,\mathrm{d}S \quad . \qquad \text{[A-9]}$$

We now consider the applied surface force terms on the right hand side of [A-9]. The viscous terms are

$$\frac{1}{V}\int_{S_{LL}(x,t) + S_{LI}(x,t) + S_{LW}(x,t)} \sigma^L_{1j} n_j\,\mathrm{d}S \quad . \qquad \text{[A-10]}$$

This integral over S_{LL} gives the normal viscous forces on the forward and rearward faces of S_L. The integral over S_{LI} is the viscous interface drag terms that are usually modeled using interface drag correlations. The integral over S_{LW} is the viscous wall shear force and is also usually modeled using drag correlations. In any case the equations used in the text generally neglect all these viscous effects (see remarks before [1]) and they are not developed further here.

We now turn to the surface pressure term on the right side of [A-9]. This term can be written as

$$\frac{1}{V}\int_{S_L(x,t)} -P_L n_1\,\mathrm{d}S = \frac{1}{V}\int_{S_{LL}(x,t)} -P_L n_1\,\mathrm{d}S + \frac{1}{V}\int_{S_{LI}(x,t) + S_{LW}(x,t)} -P_L n_1\,\mathrm{d}S \quad . \qquad \text{[A-11]}$$

Using the steps between [A-2] and [A-3] the first term reduces to

$$-\frac{\partial}{\partial x}(\alpha_L \overline{\mathrm{P}}_L) \quad . \qquad \text{[A-12]}$$

The second term in [A-11] denoted by I can be reduced further. Let P_{LI} be the average interface pressure on S_{LI} then using mean value theorem one obtains

$$I = P_{LI}\left[\frac{1}{V}\int_{S_{LI}(x,t) + S_{LW}(x,t)} -n_1\,\mathrm{d}S\right] \qquad \text{[A-13]}$$

if we remember that n_1 is zero in S_{LW}. To the above formula for I we add and subtract

$$P_{LI}\left[\frac{1}{V}\int_{S_{LL}(x,t)}-n_1\,\mathrm{d}S\right]$$

to obtain

$$I = P_{LI}\left[\frac{1}{V}\int_{S_L(x,t)}-n_1\,\mathrm{d}S+\frac{1}{V}\int_{S_{LL}(x,t)}n_1\,\mathrm{d}S\right] \quad . \qquad \text{[A-14]}$$

If the divergence theorem is applied to the first term on the right hand side of [A-14] we see that this term is zero, i.e.

$$\int_{S_L(x,t)}-n_1\,\mathrm{d}S=\int_{V_{L}(x,t)}\frac{\partial}{\partial x}(-1)\,\mathrm{d}V=0 \quad .$$

We now apply the steps between [A-2] and [A-3] again to the remaining term in [A-14] to obtain

$$I = P_{LI}\frac{\partial\alpha_L}{\partial x} \quad . \qquad \text{[A-15]}$$

Combining the above evaluations for the right hand side of [A-9] we can write the spacially average momentum equation as

$$\frac{\partial}{\partial t}(\alpha_L\overline{\rho_L u_L})+\frac{\partial}{\partial x}(\alpha_L\overline{\rho_L u_{L1}u_{L1}}) = -\frac{\partial\alpha_L\bar{P}_L}{\partial x}+P_{L1}\frac{\partial\alpha_L}{\partial x} \quad . \qquad \text{[A-16]}$$

For the incompressible case considered in the text [A-16] reduces to

$$\frac{\partial}{\partial t}(\alpha_L\rho_L v_L)+\frac{\partial}{\partial x}(\alpha_L\rho_L\overline{u_{L1}u_{L1}}) = -\frac{\partial\alpha_L P_L}{\partial x}+P_{L1}\frac{\partial\alpha_L}{\partial x} \quad , \qquad \text{[A-17]}$$

where P_L now denotes the liquid phase average pressure and v_L is the average phasic velocity. By standard arguments it is known that

$$\overline{u_{L1}u_{L1}} = v_L v_L+\overline{v_L' v_L'} \quad , \qquad \text{[A-18]}$$

where $v_L' = u_{L1} - \overline{u_{L1}}$; hence [A-17] reduces to

$$\frac{\partial}{\partial t}(\alpha_L\rho_L v_L)+\frac{\partial\alpha_L\rho_L v_L}{\partial x}+\partial\alpha_L\rho_L\overline{v_L' v_L'} = -\frac{\partial\alpha_L P_L}{\partial x}+P_{L1}\frac{\partial\alpha_L}{\partial x} \quad . \qquad \text{[A-19]}$$

Using the mass balance [A-6] to simplify the first two terms in [A-19] results in [4] of the text. This completes the derivation of the spacially averaged momentum equations for the assumptions stated in the text.

A few remarks on averaging may be appropriate at this point. Whether one performs spacial averaging (as one always must do to get a one-dimensional model), temporal averaging, or joint spacial temporal averaging one obtains formally the same averaged equation. The various terms are subject to different meanings, especially the cross-correlation terms. The closure models used by various authors in single-phase flow tend to blur the various meanings in the closure step. For the case considered here, incompressible fluid with no mass transfer the relationship between spacial and time averages can be clearly elaborated.

In the exact analysis of the two-dimensional case (see appendix B) the only wave propagation present in the unaveraged situation is the kinematic waves with speed V_M given as the real part of the expression in [B-5]. Since there is only one mechanism of dynamic wave propagation all wave lengths λ are related to their corresponding frequencies f by

$$f\lambda = V_M \quad , \qquad \text{[A-20]}$$

at least in the case of a small amplitude linear analysis. This being the case the above averaging over a spacial volume of length L clearly removes all wavelength phenomena for wavelengths less than L. Equation [A-20] then shows that this spacial averaging also removes all waves with frequency larger than $f = V_M/L$. The above spacial averaging will also "damp" or "filter" wavelength phenomena of the order $1L$ to $3L$. The corresponding frequencies are also damped. Wavelengths larger than $3L$ or $4L$ are clearly preserved in the averaged equation hence from [A-20] frequencies lower than $V_M/6L$ will also be well modeled in our spacially averaged equation. If there were multiple dynamic phenomena present in the basic unaveraged equations then the relationship between wavelength and frequency is dependent upon what phenomena are being examined and the clear one-to-one relationship of [A-20] does not hold. In this case more care is needed when comparing the spacially averaged equations with their temporally averaged counterparts.

APPENDIX B

Stability analysis for the two-dimensional Helmholtz problem between confined plates

The conventional inviscid equations of multidimensional fluid dynamics are used in this appendix to analyze the behavior of separated two-phase flow between parallel plates. The validity of the basic equations (for inviscid flow) is well established, so that results of the analysis can be used to check the predictions of the one-dimensional, two-phase equations if the frictional effects are neglected.

As in the text the fluids are assumed incompressible. The motion of small perturbations about a uniform steady state solution is examined. The necessary mathematical development differs only slightly from that in Article 232 of Lamb (1945) and hence will not be given in detail. Instead, only the modifications of Lamb's development which are necessary to treat the present case are indicated.

The modification is to allow for the finite depth of the two fluids by replacing the factors e^{ky} and e^{-ky} in Lamb's equation [5] by the factors $\cosh\,[k(y + H\alpha_G)]$ and $\cosh\,[k(y - H\alpha_L)]$, respectively. The equations which then result for the axial velocities, the interface height, and the dispersion relationship are (in the notation of the present paper)

$$u_G = v_G + H\alpha_0 i\,(\omega + v_G k)\frac{\cosh\,[k(y - H\alpha_G)]}{\sinh\,(kH\alpha_G)}e^{i(\omega t + kx)} \quad , \qquad \text{[B-1]}$$

$$u_L = v_L - H\alpha_0 i\,(\omega + v_L k)\frac{\cosh\,[k(y + H\alpha_L)]}{\sinh\,(kH\alpha_L)}e^{i(\omega t + kx)} \quad , \qquad \text{[B-2]}$$

$$y = H\alpha_0\, e^{i(\omega t + kx)} \quad , \qquad \text{[B-3]}$$

$$\rho_G \coth(kH\alpha_G)\,(\omega + v_G k)^2 + \rho_L \coth(kH\alpha_L)\,(\omega + v_L k)^2 = 0 \quad . \qquad \text{[B-4]}$$

Equations [B-1] through [B-4] determine the exact solution for the physical problem under consideration. The explicit solution of the dispersion relationship [B-4] for ω yields.

$$\frac{\omega}{k} = -\left(\frac{\rho_G^* v_G + \rho_L^* v_L}{\rho_G^* + \rho_L^*}\right) \pm \sqrt{-\frac{\rho_L^* \rho_2^* \quad (v_G - v_L)^2}{(\rho_L^* + \rho_G^*)^2}} \quad , \qquad \text{[B-5]}$$

where

$$\rho_G^* = \rho_G \coth\,(kH\alpha_G), \qquad \rho_L^* = \rho_L \coth(kH\alpha_L). \qquad \text{[B-6]}$$